Ph.D. Qualifying Exam: Differential Geometry
August 2015

Name:

Note: Please use English for your answers.

1. [20 pts] Suppose M to be a smooth manifold without boundary and N to be a smooth manifold with boundary. Prove or disprove that
   (a) $M \times N$ is a smooth manifold with boundary;
   (b) $N \times N$ is a smooth manifold with boundary.

2. [15 pts] Let $F : \mathbb{R}^n \to \mathbb{RP}^n$ be the map defined by
$$F(x_1, x_2, \ldots, x_n) = [x_1 x_2 \ldots, x_n, 1].$$
Show it is a diffeomorphism onto a dense subset of $\mathbb{RP}^n$.

3. [15 pts] Let M be a smooth manifold. If K is a closed subset of M, shows there exists a non-negative smooth function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = M$.

4. [15 pts] Let $K \subset \mathbb{R}^2$ be the boundary of the square with center $(0,0)$ and sides of length equal to 2 and $C = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$. Show there exists a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(C) = K$ and there is no diffeomorphism with the same property.

5. [10 pts] Let M be a smooth manifold without boundary, N a smooth manifold with boundary, and $F : M \to N$ a smooth map. Show that if at $p \in M$ the differential dF_p is non-singular then $F(p)$ is a point contained in the interior of N.

6. [10 pts] Show that every compactly supported smooth vector field on a smooth manifold is complete.

7. [15 pts] Let G be a Lie group.
   (a) Let $m : G \times G \to G$ denote the multiplication map. Show that the differential $dm_{(e,e)} : T_e(G \times G) \to T_eG$ is given by
$$dm_{(e,e)}(X, Y) = X + Y.$$
   (b) Let $i : G \to G$ denote the inversion map. Show that the differential $di_e : T_eG \to T_eG$ is given by
$$di_e(X) = -X.$$

THE END

Ph.D. Qualifying Exam: Algebraic Topology I
August 2015

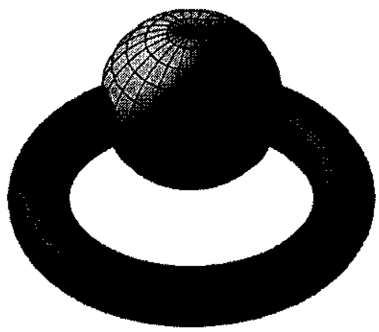
Name:

Note: Please use English for your answers.

1. Let X be the 1-skeleton of a 3-simplex and let $Y \to X$ be a 2-fold covering map such that Y is connected.
   (a) [5 pts] Find the Euler characteristic of Y.
   (b) [5 pts] Describe $\pi_1(Y)$.
   (c) [10 pts] Find two different homeomorphism types for Y and for each of them describe a covering map $Y \to X$

2. [15 pts] Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

3. As shown in the figure the torus T and the sphere S is intersecting in two disjoint circles. Let $X = S \cup T$.



   (a) [10 pts] Find a finite cell-complex structure on X.
   (b) [15 pts] Compute $H_i(X)$, for $i = 1, 2$.

4. Let $0 \to \mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$ be a short exact sequence of chain complexes $\mathcal{A} = \{A_n, \partial\}$, $\mathcal{B} = \{B_n, \partial\}$, $\mathcal{C} = \{C_n, \partial\}$ and chain maps ϕ and ψ.
   (a) [10 pts] Construct the connecting homomorphisms $\partial_* : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$ making the sequence

$$\to H_{n+1}(\mathcal{C}) \xrightarrow{\partial_*} H_n(\mathcal{A}) \xrightarrow{\phi_*} H_n(\mathcal{B}) \xrightarrow{\psi_*} H_n(\mathcal{C}) \xrightarrow{\partial_*} H_{n-1}(\mathcal{A}) \xrightarrow{\phi_*} H_{n-1}(\mathcal{B}) \to$$

   exact.
   (b) [15 pts] Show that the above sequence is exact.

5. [15 pts] Let $A \subset X$ and let $X \cup CA$ denote the mapping cone of the inclusion map $A \hookrightarrow X$. Prove that $H_n(X \cup CA, X) \approx \widetilde{H}_{n-1}(A)$.

THE END

Ph.D. Qualifying Exam: Real Analysis
August 2015

Name:

Note: Please use English for your answers.

1. Compute the following limit and justify the calculation:

   (a) [5 pts]
   $$\sum_{n=0}^{\infty} \int_0^{\pi/4} (1 - \sqrt{\sin x})^n \cos x \, dx.$$

   (b) [10 pts]
   $$\lim_{n\to\infty} \int_0^{\infty} \frac{\sin(x^n)}{x^n} \, dx.$$

2. [15 pts] We denote $\mathcal{L}$ the class of Lebesgue measurable sets and m the Lebesgue measure. Prove the following: If $E \in \mathcal{L}$ and $m(E) > 0$, then for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

3. [15 pts] Let $(X, \mathcal{M}, \mu)$ be a σ-finite positive measure space and $\{\nu_n\}$ be a sequence of σ-finite positive measures on $\mathcal{M}$ with $\nu_n \ll m$ for all $n \in \mathbb{N}$ such that
   $$\nu_n(E) \leq \nu_{n+1}(E), \quad \text{for all } n \in \mathbb{N},\ E \in \mathcal{M}.$$
   Define $\nu(E) = \lim_{n\to\infty} \nu_n(E)$ for each $E \in \mathcal{M}$. Show that ν defines a measure on $\mathcal{M}$ and ν satisfies $\nu \ll \mu$. And express $\frac{d\nu}{d\mu}$ in terms of ν_n.

4. [20 pts] Let X and Y be Banach spaces. Let T be a surjective bounded linear map from X to Y. Show that T is open.

5. Prove the following statement:

   (a) [10 pts] If A is any set and $0 < p < q \leq \infty$, then $l^p(A) \subset l^q(A)$.

   (b) [10 pts] Let $(X, \mathcal{M}, \mu)$ be a measure space. If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^q(\mu) \subset L^p(\mu)$.

6. [15 pts] Let $f_n, f \in L^p[0,1]$ with $1 \leq p < \infty$. Suppose that $f_n \to f$ pointwise. Prove that $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.

THE END

Ph.D. Qualifying Exam: Complex Analysis
August 2015

Name:

Note: Please use English for your answers.

1. [15 pts] State and give a proof of the Argument Principle.
2. [15 pts] State and give a proof of the Rouché Theorem.
3. [10 pts] Explain why each of the polynomials $p(z) = 2z^{10} + 4z^2 + 1$ and $q(z) = 2z^{10} - 4z^2 + 1$ has exactly two zeroes in the disk $|z| < 1$.
4. [15 pts] Determine the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ integrating
$$f(z) = \frac{\pi \cot \pi z}{z^2}$$
on an appropriate domain.
5. [15 pts] Find the Hadamard factorization of the function $e^z - 1$.
6. [15 pts] Explain why the function:
$$f(z) := \int_0^z \frac{d\xi}{(1-\xi^2)^{\frac{1}{2}}}$$
maps the upper half-plane $\mathbb{H}$ conformally to the strip
$$S := \{x + iy \in \mathbb{C}, |x| < \frac{\pi}{2}, y > 0\}.$$
7. [15 pts] Show that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disk $\mathbb{D}$ take the form
$$e^{i\theta} \frac{z-\beta}{z-\overline{\beta}},$$
with $\theta \in \mathbb{R}$, $\beta \in \mathbb{H}$.

THE END

Ph.D. Qualifying Exam: Advanced Statistics
August 2015

Name:

Note: Please use English for your answers.

1. Let $X_1, \cdots, X_n$ be a random sample from Exponential distribution $Exp(\lambda)$ with pdf $f(x;\lambda) = \lambda^{-1}e^{-x/\lambda}$ and let $X_{(i)}$ be the ith smallest of the sample.

   (a) [5 pts] Find $E(X_{(1)})$.

   (b) [7 pts] Let $S_j = \sum_{i=1}^{j} X_i$. Find the distribution of S_k/S_n, $1 \le k < n$.

   (c) [12 pts] Let $n = 3$. Find the best unbiased estimator of $P(X_{(1)} \ge 1)$ if it exists.

2. [10 pts] Let $X_1, \cdots, X_n$ be a random sample from $f(x;\theta)$, where $f(x;\theta)$ satisfies the condition of the Cramér-Rao Theorem. Let $W = W(X_1, \cdots, X_n)$ is an unbiased estimator of $\tau(\theta)$. Prove that $Var(W)$ attains the Cramér-Rao Lower Bound if and only if W is a linear function of $\frac{\partial}{\partial\theta} \log f(X_1, \cdots, X_n; \theta)$.

3. [10 pts] Let $X_1, \cdots, X_n$ be a random sample from $f(x;\theta)$. Let T_1 be a minimal sufficient statistic and T_2 a sufficient statistic for θ. Let W be an unbiased estimator of θ. Define $U_i = E(W|T_i)$, $i = 1, 2$.

   Compare $Var(U_1)$ with $Var(U_2)$. Which is larger? Why?

4. Let X_1, X_2 be independent with $X_i \sim N(0, \sigma_i^2)$ where σ_i's ($\sigma_1 < \sigma_2$) are known. Let Λ be a Bernoulli random variable with probability of success θ. Define

$$Y = \Lambda X_1 + (1 - \Lambda) X_2. \quad (1)$$

   Y is generated in two steps: First, generate X_1, X_2, and Λ; Then obtain Y by the formula (1).

   Suppose we have a random sample $Y_1, \cdots, Y_n$ from the distribution of Y and we don't have any observations on X_1, X_2, and Λ. In other words, we have observations only on Y_i's.

   Denote the pdf of $N(0, \sigma_i^2)$ by f_i, $i = 1, 2$. Note that the MLE $\hat{\theta}$ of θ is not given in a closed form.

   (a) [3 pts] Write down the log-likelihood function of θ.

   (b) [3 pts] Derive an estimating equation for $\hat{\theta}$. [The estimating equation is given by ∂(log-likelihood function)$/\partial\theta = 0$.]

   (c) [10 pts] What is the condition so that the MLE $\hat{\theta}$ lies in the interval $(0, 1)$?

– Over –

5. Let $X_1, \cdots, X_n$ be iid Bernoulli random variables with probability of success θ. Let $T = \sum_{i=1}^n X_i$.

   (a) [10 pts] For $k < n$, find the best unbiased estimator of $P(X_1 X_2 \cdots X_k = 1)$ if it exists.

   (b) [4 pts] Find the MGF of T.

   (c) [10 pts] Find the limiting distribution of $\sqrt{n}(T/n - \theta)$.

6. Let $X_1, \cdots, X_n$ be a random sample and let $\bar{X}$ and S^2 be the sample mean and the sample variance respectively. Denote $\theta_1 = E(X_1)$, $\theta_j = E(X_1 - \theta_1)^j$, $j = 2, 3, \cdots$.

   (a) [3 pts] Show that

$$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - X_j)^2.$$

   (b) [5 pts] Find $Cov(\bar{X}, S^2)$ in terms of θ_j, $j = 1, 2, 3, 4$.

   (c) [8 pts] Find $Var(S^2)$ in terms of θ_j, $j = 1, 2, 3, 4$.

THE END

Ph.D. Qualifying Exam: Combinatorics
August 2015

Name:

Note: Please use English for your answers.

1. (20 pts) Let $A_1, A_2, \ldots, A_m$ be distinct subsets of $\{1, 2, \ldots, n\}$. Prove that if $|A_i \cup A_j|$ is odd for all $i \neq j$, then $m \leq 2^{\lfloor n/2 \rfloor}$.

2. (20 pts) Prove that there exists N such that every set of N distinct points in $\mathbb{R}^2$ having no three points in a line must contain 10 points which are the vertices of a convex 10-gon.

3. (20 pts) Prove that there exists $c > 0$ such that every set of N positive integers admits a subset B of more than cN integers such that $x + y + z \notin B$ for all $x, y, z \in B$.

4. (20 pts) Let $R(t, t, t)$ be the minimum n such that every coloring of the edges of the complete graph K_n with 3 colors induces a monochromatic complete subgraph consisting of t vertices. Prove that $R(t, t, t) > 3^{t/2}$ for all $t > 4$.

5. (20 pts) Let $a_1, a_2, \ldots, a_m$ be distinct points in the n-sphere S^n in $\mathbb{R}^{n+1}$. Prove that if there exist $d_1, d_2, d_3 \in \mathbb{R}$ such that $\|a_i - a_j\| \in \{d_1, d_2, d_3\}$ for all $i \neq j$, then $m \leq \binom{n+1}{3}$.

THE END